Controlling the unstable steady state in a multimode laser

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A method to control unstable steady states in high-dimensional systems is described and implemented on a model of a multimode laser with an intracavity doubling crystal. Our control method uses the duration of time for which feedback control is applied as an addition parameter. This is necessary to account for control activated transients in a high-dimensional system.

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The use of a system's natural dynamics to force the system into a desired unstable state and thus achieve control offers an advantage over classical control methods. By applying small amplitude feedback to a readily available system parameter so that the system evolves towards the desired state, difficult or costly modifications to the system that alter its dynamics are unnecessary. This idea is apparent in a method to stabilize periodic orbits originated by Ott, Grebogi, and York [1]. They used linear control theory and feedback to an available system parameter to direct the system to the stable manifold of the unstable state. Ideally, control could then be turned off as the natural dynamics along the stable manifold continued to contract the system towards the desired state. Additionally, information about the local dynamics of the desired unstable state that is required to formulate the control law could be obtained by reconstructing the system's phase space from experimental data [2]. The control method is said to be "model independent" as a detailed model of the system does not have to be constructed (see Ref. [3] for a review).

A limitation of the Ott-Grebogi-Yorke control method occurs in high-dimensional systems where control perturbations induce transients off the unstable manifold that hinder the effectiveness of the method. It is interesting then that a related scalar control method, called occasional proportional feedback (OPF) [4], has been successful in controlling the steady state and periodic orbits of a multimode laser, which is a high-dimensional system [5,6]. However, the OPF feedback method also requires the careful tuning of additional experimental parameters. In particular, the laser control experiment requires adjustment of control perturbation pulse width. This observation motivated us to develop a control method that explicitly uses the duration of time the control signal is applied, called control duration, as an additional feedback parameter to control steady states in high-dimensional systems [7]. An important difference between OPF and our method is that while in the case of OPF the control pulse width is fixed, we use the pulse width, or as we call it, the control duration, as an additional feedback parameter.

After briefly reviewing the theory of our control method [7], we describe its numerical application to the multimode laser system used in the OPF control experiments. The unstable manifold is a focus, while the stable manifold is four dimensional with two pairs of complex-conjugate eigenvalues. Using the control duration as an additional parameter compensates for control activated transients that occur on the stable manifold.

Consider the general system

$$\frac{d\mathbf{z}}{dt} = \mathbf{F}(\mathbf{z}, P), \tag{1}$$

where $\mathbf{z}$ is an n-dimensional state variable, P is a scalar parameter of the system and will be used as the control variable, and $\mathbf{F}$ is a nonlinear vector function of the state and control variables. We assume the existence of a steady state solution given by $(\mathbf{z}(P), P)$. We wish to establish control about a particular steady state when $P = \bar{P}$ and $\mathbf{z}(\bar{P}) = \bar{\mathbf{z}}$. To this end we approximate the dynamics about this steady state point as

$$\frac{d\mathbf{x}}{dt} = \mathbf{A} \cdot \mathbf{x} + \mathbf{B}p, \tag{2a}$$

$$\mathbf{A} = D_z\mathbf{F}(\bar{\mathbf{z}}, \bar{P}), \quad \mathbf{B} = \frac{d\mathbf{F}(\bar{\mathbf{z}}, \bar{P})}{dP}, \tag{2b}$$

where $\mathbf{x}$ and p are small deviations ($\mathbf{x} = \mathbf{z} - \bar{\mathbf{z}} \ll 1$ and $p = P - \bar{P} \ll 1$) from the steady state values $\bar{\mathbf{z}}$ and $\bar{P}$, respectively. We assume that there is a single complex-conjugate pair of unstable eigenvalues $\sigma_u(p) \pm i\omega(p)$, where $\sigma_u > 0$ and $d\sigma_u/dp \neq 0$. This implies the existence of a Hopf bifurcation for some lower value of the parameter P. The only restriction on the additional eigenvalues is that they have a negative real part. For model-independent control, the parameters of (2a), i.e., $\mathbf{A}$ and $\mathbf{B}$, can be determined by embedding the flow of the physical system into an artificial phase space [2].

Provided that $\mathbf{A}$ is nonsingular, i.e., P is not a bifurcation point, then the general solution to (2a) is

$$\mathbf{x}(t) = e^{\mathbf{A}t} \cdot [\mathbf{x}(0) + \mathbf{A}^{-1} \cdot \mathbf{B}p] - \mathbf{A}^{-1} \cdot \mathbf{B}p. \tag{3}$$

The $n \times n$ matrix $\mathbf{A}$ can be block diagonalized as $\mathbf{A} = \mathbf{S} \cdot \Lambda \cdot \mathbf{S}^{-1}$, where $\mathbf{S}$ is composed of the right eigenvectors $\mathbf{e}_i, i = 1, \ldots, n$, and $\mathbf{S}^{-1}$ is composed of the left eigenvectors $\mathbf{f}_i, i = 1, \ldots, n$. We designate the right eigenvectors associated with the unstable complex-conjugate pair of eigenvalues as $\mathbf{e}_1$ and $\mathbf{e}_2$ (similarly, $\mathbf{f}_1$ and $\mathbf{f}_2$ are the corresponding left eigenvectors).

The goal of the control method is that given some initial error $\mathbf{x}(0)$, we determine the parameter variation p and the control duration $T_c = (2\pi q)/\omega$ (q is the unknown) such that $\mathbf{x}(t) \to 0$ as $t \to \infty$. This is accomplished by forcing the system state to lie entirely within the stable manifold at the end of the control period, specifically,

$$\mathbf{x}(T_c) = \mathbf{S} \cdot \mathbf{k} \text{ where } \mathbf{k} = (0, 0, k_3, \ldots, k_n). \quad (4)$$

Substituting $t = T_c$ into (3), we obtain n equations for the n unknowns $p, q, k_3, ..., k_n$. We are concerned only with the first two equations that determine p and q. Once these are determined the remaining equations determine only the k_i. We allow these to be arbitrary because they specify only the location of the system in the stable manifold. Once the dynamics is on the stable manifold, the system will then evolve towards the steady state.

Solving the first two equations of (3) for p, we obtain

$$p = \frac{-\lambda(q)\mathbf{f}_1 \cdot \mathbf{x}(0)}{d_1[\lambda(q) - \cos(2\pi q)] + d_2 \sin(2\pi q)}, \quad (5a)$$

$$\lambda(q) = e^{2\pi q \sigma_u/\omega}, \quad (5b)$$

$$d_1 = \frac{(\sigma_u \mathbf{f}_1 - \omega \mathbf{f}_2) \cdot \mathbf{B}}{\sigma_u^2 + \omega^2}, \quad (5c)$$

$$d_1 = \frac{(\omega \mathbf{f}_1 + \sigma_u \mathbf{f}_2) \cdot \mathbf{B}}{\sigma_u^2 + \omega^2} \quad (5d)$$

and the following transcendental equation for q:

$$[\lambda(q) - \cos(2\pi q)](d_1 \mathbf{f}_2 - d_2 \mathbf{f}_1) + \sin(2\pi q)(d_1 \mathbf{f}_1 + d_2 \mathbf{f}_2) \cdot \mathbf{x}(0) = 0. \quad (6)$$

The solution to (6) is multivalued and we take $q \in (0, 1)$ so that control is applied for less than one natural period of the system.

We reiterate that the use of the control duration as a second parameter compensates for the deviations off the unstable manifold into the stable subspace due to the control perturbation. It is not used to formulate a "multiparameter scheme" [8] due to the two-dimensional unstable manifold. If we had considered a simple two-dimensional system that had undergone a Hopf bifurcation, the two-dimensional unstable manifold or focus would be controllable using only amplitude perturbations of p [7].

To control the system we have the following algorithm. Given $\mathbf{x}(0)$, (6) is solved numerically to determine q. Next, p is determined using (5a). Using these values, $\mathbf{x}(T_c)$ will lie in the stable manifold and eventually decay to 0. In a real system, noise and small errors will require the system to be monitored and control reapplied.

We now apply our method to the control of the unstable steady state of a multimode Nd:YAG laser (where YAG denotes yttrium aluminum garnet) with an intracavity potassium titanyl phosphate (KTP) frequency-doubling crystal. We consider the case of three couple modes with each being modeled by rate equations for the mode intensity I_j and the gain G_j [9]:

$$\tau_c \frac{dI_j}{dt} = I_j \left(G_j - \alpha - \epsilon (g I_j + 2 \sum_{k \neq j} \mu_{jk} I_k) \right), \quad (7a)$$

$$\tau_f \frac{dG_j}{dt} = \gamma - G_j \left(1 + I_j + \beta \sum_{k \neq j} I_k \right). \quad (7b)$$

In these equations, τ_c is the cavity round-trip time (0.2 ns), τ_f is the fluorescence lifetime (240 μs), and α is the nondimensional loss (0.01). The geometrical factor g (0.01) depends on the orientation of the YAG and KTP crystals. Each mode can be polarized in either of two orthogonal directions so that $\mu_{jk} = g$ when modes j and k are of the same polarization and $\mu_{jk} = 1 - g$ when they are in different polarizations; our simulations consider the case when modes 1 and 2 are in the same polarization and both are orthogonal to mode 3. The conversion efficiency of infrared light to frequency-doubled green light is indicated by the parameter ϵ (5.0×10^{-6}), while β is the cross saturation parameter related to the competition among the different modes (0.6). Finally, γ is the small signal gain, which is related to the pump intensity, and will serve as the control parameter.

We chose to control the unstable steady state when $\gamma = 0.5$ by adding small perturbations p for a calculated duration of time $T_c(q)$ as determined in Eqs. (5a) and (6). The parameters required to determine p and q are calculated using Eqs. (7a) and (7b), but could be made model independent by using an embedded times series from experimental data. The error vector $\mathbf{x} = (I_1, G_1, \ldots) - (\bar{I}_1, \bar{G}_1, \ldots)$ is measured at a specified sampling rate and the proper correction to p and q is determined. Additionally, we specify that the error vector be at least $|\mathbf{x}| > 1.0^{-6}$ before control is activated; we call this the control criteria. This was done only so the growth in the error could be seen in the figures and is not necessary, in general.

Figure 1 shows the controlled steady state for approximately 40 ms, during which time there are 200 control pulses. The control is then turned off so that the system evolves to antiphase oscillations (also known as

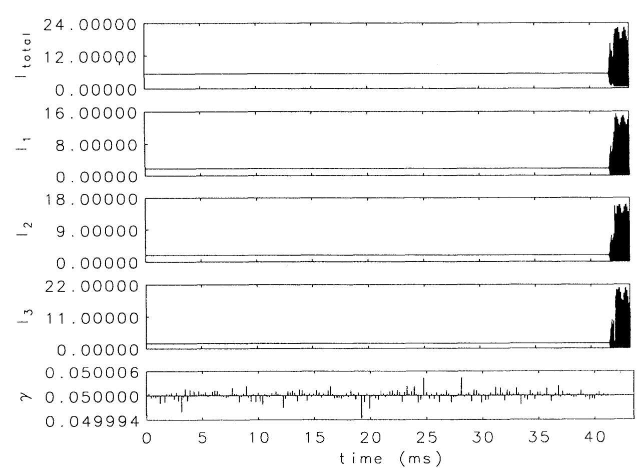

FIG. 1. The laser is controlled for approximately 40 ms at which time the feedback control is turned off and the system evolves to antiphase oscillations. The total intensity output of the laser is the sum of the individual mode intensities $I_{\text{total}} = I_1 + I_2 + I_3$.

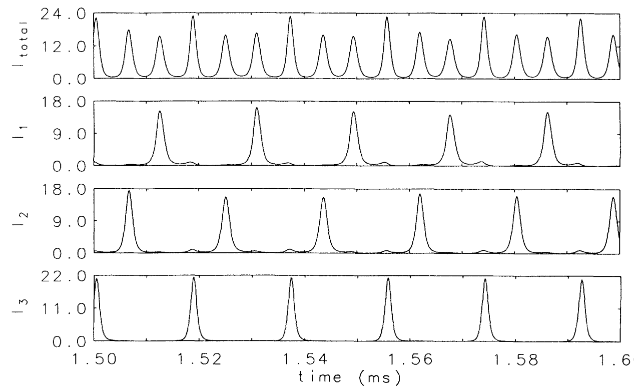

FIG. 2. Left uncontrolled, any deviations from the unstable steady state evolve to antiphase oscillations ordered mode 1,3,2. "Antiphase" refers to the fact that each oscillator is phase shifted by 1/3 of the full period from its neighbor.

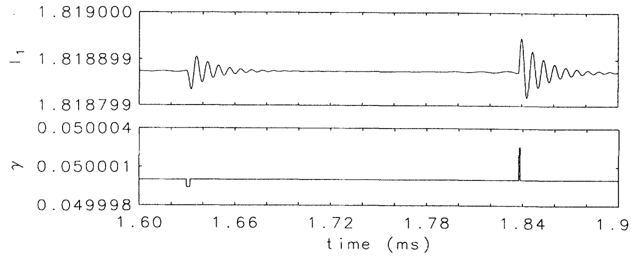

FIG. 3. Detailed view of the control action on the laser. Variations in the parameter q result in control pulses of different duration.

splay-phase oscillations) [9] characteristic of the multimode laser (see Fig. 2).

The action of a single control pulse in γ can be seen in Fig. 3. The control pulse at 1.63 ms directs the system onto the stable manifold of the unstable steady state indicated by the decaying oscillations that follow. Small errors or noise are amplified by the unstable dynamics so that there is slow growth in the oscillations away from the steady state. Left uncontrolled the system would evolve to the antiphase oscillations shown in Fig. 2. However, upon resampling of the system at approximately 1.84 ms, control was reactivated to bring the system back to the unstable steady state.

Note that the pulse width of the two major control corrections is visibly different; the correct control duration, or q, was determined using (6). During the simulation shown in Fig. 1, it was found that $q \in (0.997, 0.014)$, the mean value was $\bar{q} = 0.384$, and the standard deviation was $\sigma_q = 0.263$. The fact that there is a low variance in q suggests a possible contributor to the success of the experiments using OPF when the control duration was held fixed.

We have also simulated the system with q fixed to the mean value $q = \bar{q}$. Control is less efficient in that the system is not as precisely placed on the stable manifold so that control must occur much more often. Alternatively, we conjecture that control may be established by fixing the p and applying feedback *only* to the control duration q. This may be advantageous in certain applications.

After the control pulses the initial error of the decaying oscillations is larger than the control criteria. The present control method is optimal in the sense of placing the system on the stable manifold, but is unable to dictate *where* on the stable manifold. [Notice in (4) that the k_i cannot be specified and are arbitrary.] It is a future challenge to optimize the control so that the size of the error fluctuations is minimized. This would require a trade-off between the accuracy of placement on the stable manifold and the distance from the unstable steady state.

Finally, if the system is sampled during the oscillations that follow a control pulse, control should be reactivated. In Fig. 1 this is prevented by disallowing a control pulse if the system is on the stable manifold. (This was done so that the statistics on q would be determined only from a correction when the system is on the unstable manifold.) However, the control method is successful even without this restriction. When the system is on, or very close to, the stable manifold and orthogonal to the unstable manifold, the control pulse is very small. Specifically, the error vector is perpendicular to the unstable left eigenvectors $(\mathbf{x} \cdot \mathbf{f}_i \approx 0, i = 1, 2)$ so that $p \approx 0$.

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